Exo 1, correction

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Abstract

The conditional expectation.

Exercice 0 Let T be a discrete random variable, and X be measurable with respect to $\sigma(T)$. Show that it exists a measurable function f such that $X = f(T)$.
Prove that the converse is also true.
(Please note that this equivalence also holds if T is not discrete.)

Proof. We will only prove the first implication.
First, note that it is enough to prove that, for every $t \in S_T$, we have $\omega_1, \omega_2 \in \{T \in t\}$, implies $X(\omega_1) = X(\omega_2)$, because in this case we can define $f(t) = X(\omega_1)$.
Thereby, suppose by absurd that for such ω_i, we have $X(\omega_1) =: a \neq b := X(\omega_2)$. Then, considere $A_1 := \{T = t\} \cap X^{-1}(a)$ and $A_2 := \{T = t\} \cap X^{-1}(b)$ in $\sigma(T)$ and disjoint in $\{T = t\}$ by assumption. But this contradicts the fact that X is measurable with respect to $\sigma(T)$. ■

I Definition

Exercice 1 Let X, T be two random variables such that $\mathbb{E}[|X|] < \infty$ and such that T is discrete. Give the definition of the conditional expectation $\mathbb{E}[X \mid T]$.
Prove that the following identity, characterizing the conditional expectation (see course) is true,

$$\forall f \ : \ \mathbb{R} \mapsto \mathbb{R} \text{ mesurable and bounded,} \qquad \mathbb{E}\left[f(T)\,\mathbb{E}[X \mid T]\right] = \mathbb{E}\left[f(T)X\right].$$

Proof. In the discrete case, we have

$$\mathbb{E}[X \mid T] = \sum_{t \in S_T} \mathbb{E}[X \mid T = t]\mathbf{1}_{T=t}.$$

Thus we can compute

$$\begin{aligned}
\mathbb{E}\left[f(T)\,\mathbb{E}[X \mid T]\right] &= \mathbb{E}\left[f(T)\sum_{t \in S_T} \mathbb{E}[X \mid T = t]\mathbf{1}_{T=t}\right] = \sum_t \mathbb{E}[X \mid T = t]\,\mathbb{E}[f(T)\mathbf{1}_{T=t}] \\
&= \sum_t \mathbb{E}[X \mid T = t] f(t)\mathbb{P}(T = t) = \sum_t \mathbb{E}[Xf(t) \mid T = t]\mathbb{P}(T = t) \\
&= \sum_t \mathbb{E}[Xf(T) \mid T = t]\mathbb{P}(T = t) = \mathbb{E}[Xf(T)],
\end{aligned}$$

the last equility coming from the totale probabilities formula. ■

Exercice 2 Let X, Y be two random variables such that (X, Y) has a density $h(x, y)$ and X admits a first moment. Let $A \in \mathcal{B}(\mathbb{R})$ and $y_0 \in \mathbb{R}$. Compute, for $\varepsilon > 0$, $\mathbb{P}(X \in A \mid Y \in]y_0 - \varepsilon; y_0 + \varepsilon[)$, and propose an expression for $\mathbb{E}[X \mid Y]$.
Finally, show that indeed

$$E[X \mid Y] = \frac{\int_{\mathbb{R}} x\, h(x, Y)dx}{\int_{\mathbb{R}} h(x, Y)dx}. \tag{I.1}$$

Proof. We have, for $y_0 \in \mathbb{R}$ and $\varepsilon > 0$, and supposing that the density if continuous, the following formal reasonning

$$\begin{aligned}\mathbb{P}(X \in A \mid Y \in]y_0-\varepsilon; y_0+\varepsilon[) &= \frac{\mathbb{P}(X \in A \ \cap \ Y \in]y_0-\varepsilon; y_0+\varepsilon[)}{\mathbb{P}(Y \in]y_0-\varepsilon; y_0+\varepsilon[)} \\ &= \frac{\int_{y_0-\varepsilon}^{y_0+\varepsilon}\int_A h(x,y)dxdy}{\int_{y_0-\varepsilon}^{y_0+\varepsilon}\int_{\mathbb{R}} h(x,y)dxdy} \\ &\longrightarrow_{\varepsilon\mapsto 0} \quad \frac{2\varepsilon\int_A h(x,y_0)dx}{2\varepsilon\int_{\mathbb{R}} h(x,y_0)dx} = \int_A \frac{h(x,y_0)}{\int_{\mathbb{R}} h(x',y_0)dx'}dx\end{aligned}$$

Thus, we can infere that the conditional law of X knowing $\{Y = y_0\}$ has a density

$$\frac{h(\cdot,y_0)}{\int_{\mathbb{R}} h(x',y_0)dx'},$$

and thus we would have indeed (**??**).

Now, we have to prove to prove rigorously (**??**).
By Exercice 0, we know that it exists f measurable such that $\mathbb{E}[X \mid Y] = f(Y)$, thus we have to find the f such that, for every measurable bounded function g we have

$$\mathbb{E}[Xg(Y)] = \mathbb{E}[f(Y)g(Y)]. \tag{I.2}$$

The left hand side checks

$$\mathbb{E}[Xg(Y)] = \iint_{\mathbb{R}\times\mathbb{R}} xg(y)h(x,y)dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} xh(x,y)dx\right)g(y)dy.$$

The right hand side checks

$$\mathbb{E}[f(Y)g(Y)] = \iint_{\mathbb{R}\times\mathbb{R}} f(y)g(y)h(x,y)dxdy = \int_{\mathbb{R}}\left(f(y)\int_{\mathbb{R}} h(x,y)dx\right)g(y)dy.$$

This being true for every function g, we can deduce (why?) that, for almost all y,

$$f(y)\int_{\mathbb{R}} h(x,y)dx = \int_{\mathbb{R}} xh(x,y)dx.$$

Thus, indeed, the f checking (**??**) is the one given by (**??**). ■

II Properties

Exercice 3 Let $X, X_1, X_2, \ldots$ be random variables such that $\mathbb{E}[|X_i|] < \infty$, and $\mathcal{F}$ be a σ-algebra. Prove that the following properties are true,

i) $\mathbb{E}[\lambda_1 X_1 + \lambda_2 X_2 \mid \mathcal{F}] = \lambda_1\,\mathbb{E}[X_1 \mid \mathcal{F}] + \lambda_2\,\mathbb{E}[X_2 \mid \mathcal{F}]$.

ii) If $X \geq 0$, then $\mathbb{E}[X \mid \mathcal{F}] \geq 0$. Deduce that if $X = 0$, then $\mathbb{E}[X \mid \mathcal{F}] = 0$.

iii) $\big|\mathbb{E}[X \mid \mathcal{F}]\big| \leq \mathbb{E}[|X| \mid \mathcal{F}]$. Deduce that $\|\mathbb{E}[X \mid \mathcal{F}]\|_1 \leq \|X\|_1$, which means that if X is integrable, then $\mathbb{E}[X \mid \mathcal{F}]$ also.

iv) If X is measurable with respect to $\mathcal{F}$, one has $\mathbb{E}[X \mid \mathcal{F}] = X$.

v) If φ is a convex function on $\mathbb{R}$ such that $\mathbb{E}[|\varphi(X)|] < \infty$, one has $\varphi\big(\mathbb{E}[X \mid \mathcal{F}]\big) \leq \mathbb{E}\left[\varphi(X) \mid \mathcal{F}\right]$ (this is Jensen conditional inequality).

vi) If $X_n \mapsto X$ *as*, and $\forall n \geq 0$, $|X_n| \leq Y \in L^1$, one has $\mathbb{E}[X_n \mid \mathcal{F}] \mapsto \mathbb{E}[X \mid \mathcal{F}]$ in L^1 (this is conditional dominated convergence theorem).

vii) If Z is a $\mathcal{F}$-measurable bounded random variable, then $\mathbb{E}[X \mid \mathcal{F}]Z = \mathbb{E}[XZ \mid \mathcal{F}]$.

viii) $\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}]\right] = \mathbb{E}[X]$.

ix) If $\mathcal{F}'$ is another σ-algebra such that $\mathcal{F}' \subset \mathcal{F}$, one has

$$\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}'] \mid \mathcal{F}\right] = \mathbb{E}[X \mid \mathcal{F}'] = \mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}] \mid \mathcal{F}'\right].$$

Proof. Points i), ii), iii) and iv) are given by the proof of the existence and unicity of the conditional expectation.

v) Let $x_0 := \mathbb{E}[X \mid \mathcal{F}]$, and by classical properties of convex functions, we know that we can define $a, b \in \mathbb{R}$ such that $ax + b \leq \varphi(x)$ and $ax_0 + b = \varphi(x_0)$. Thus

$$\varphi\big(\mathbb{E}[X \mid \mathcal{F}]\big) = \varphi(x_0) = ax_0 + b = a\,\mathbb{E}[X \mid \mathcal{F}] + b = \mathbb{E}[aX + b \mid \mathcal{F}] \leq \mathbb{E}[\varphi(X) \mid \mathcal{F}].$$

Note that it is just the classical proof of Jensen's inequality, written for conditional expectations. Also note that x_0, a and b are random variables, but since they are $\mathcal{F}$-measurable the computations we made are allowed.

vi) By the classical dominated convergence, we have $\|X_n - X\|_1 \mapsto 0$. Thus by i) and then by iii) we have indeed

$$\|\,\mathbb{E}[X_n \mid \mathcal{F}] - \mathbb{E}[X \mid \mathcal{F}]\|_1 = \|\,\mathbb{E}[X_n - X \mid \mathcal{F}]\|_1 \leq \|X_n - X\|_1 \mapsto 0.$$

vii) Take $A \in \mathcal{F}$, and notice that $Z\mathbf{1}_A$ is $\mathcal{F}$-measurable, thus by definition indeed

$$\mathbb{E}\left[(\mathbb{E}[X \mid \mathcal{F}]Z)\mathbf{1}_A\right] = \mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}](Z\mathbf{1}_A)\right] = \mathbb{E}[XZ\mathbf{1}_A].$$

viii) It's just the definition of the conditional expectation with $\mathbf{1}_A = 1$, *ie* $A = \Omega$.

ix) For clarity, note $Z := \mathbb{E}[X \mid \mathcal{F}']$.
The first equality is true because Z is $\mathcal{F}$-measurable and thus we can use point iv).
The second equality is a little bit more tricky. We have to show that for every $F \in \mathcal{F}'$,

$$\mathbb{E}[Z\mathbf{1}_F] = \mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}]\mathbf{1}_F\right].$$

But since $\mathbf{1}_F$ is $\mathcal{F}$-measurable, we can use points vii) and viii) to show that

$$\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}]\mathbf{1}_F\right] = \mathbb{E}\left[\mathbb{E}[X\mathbf{1}_F \mid \mathcal{F}]\right] = \mathbb{E}[X\mathbf{1}_F].$$

Finally, one can note that $\mathbb{E}[X\mathbf{1}_F] = \mathbb{E}[Z\mathbf{1}_F]$ by the definition of Z.

∎

Exercice 4 Let X_1, X_2 be two random variables such that $\mathbb{E}[|X_i|] < \infty$, and $\mathcal{F}$ be a σ-algebra. Suppose that X_1 is independant of $\mathcal{F}$, and show that

$$\mathbb{E}[X_1 \mid \mathcal{F}] = \mathbb{E}[X_1] \ as.$$

Suppose that X_1 is independant of $\sigma(\sigma(X_2), \mathcal{F})$, and that $\mathbb{E}[|X_1X_2|] < \infty$, and show that

$$\mathbb{E}[X_1X_2 \mid \mathcal{F}] = \mathbb{E}[X_1 \mid \mathcal{F}]\,\mathbb{E}[X_2 \mid \mathcal{F}].$$

Proof.

- First, we have to show that, for every $A \in \mathcal{F}$,
$$\mathbb{E}\left[\mathbb{E}[X_1]\mathbf{1}_A\right] = \mathbb{E}[X_1\mathbf{1}_A].$$
And indeed thanks to independance, we have
$$\mathbb{E}\left[\mathbb{E}[X_1]\mathbf{1}_A\right] = \mathbb{E}[X_1]\,\mathbb{E}[\mathbf{1}_A] = \mathbb{E}[X_1\mathbf{1}_A].$$

- Now, we have to show that, for every $A \in \mathcal{F}$,
$$\mathbb{E}\left[\mathbb{E}[X_1 \mid \mathcal{F}]\,\mathbb{E}[X_2 \mid \mathcal{F}]\mathbf{1}_A\right] = \mathbb{E}[X_1X_2\mathbf{1}_A].$$
And indeed thanks to independance, we have, noting $Z := \mathbb{E}[X_2 \mid \mathcal{F}]\mathbf{1}_A = \mathbb{E}[X_2\mathbf{1}_A \mid \mathcal{F}]$ a $\mathcal{F}$-measurable function
$$\begin{aligned}\mathbb{E}\left[\mathbb{E}[X_1 \mid \mathcal{F}]\,\mathbb{E}[X_2 \mid \mathcal{F}]\mathbf{1}_A\right] &:= \mathbb{E}\left[\mathbb{E}[X_1 \mid \mathcal{F}]Z\right] = \mathbb{E}[X_1Z] \\ &= \mathbb{E}[X_1\,\mathbb{E}[X_2\mathbf{1}_A \mid \mathcal{F}]] = \mathbb{E}[X_1]\,\mathbb{E}[\mathbb{E}[X_2\mathbf{1}_A \mid \mathcal{F}]] \\ &= \mathbb{E}[X_1]\,\mathbb{E}[X_2\mathbf{1}_A] = \mathbb{E}[X_1X_2\mathbf{1}_A]\end{aligned}$$

∎

III Exercices

Exercice 5 Let X, Y, Z be three random variables with a first order such that (X, Z) has the same law as (Y, Z). Show that for all $f \geq 0$ measurable and bounded,
$$\mathbb{E}[f(X) \mid Z] = \mathbb{E}[f(Y) \mid Z].$$
Then, let $g \geq 0$ a measurable and bounded function, and define $h_1(X) := \mathbb{E}[g(Z) \mid X]$ and $h_2(Y) := \mathbb{E}[g(Z) \mid Y]$. Show that $h_1 = h_2$, μ-*ae*, where μ is the law of X (and of Y).

Proof. For the first part, note that for every g measurable and bounded, since the equality of the laws
$$\begin{aligned} & \mathbb{E}[f(X)g(Z)] = \mathbb{E}[f(Y)g(Z)] \\ \Leftrightarrow \quad & \mathbb{E}\left[\mathbb{E}[f(X) \mid Z]g(Z)\right] = \mathbb{E}\left[\mathbb{E}[f(Y) \mid Z]g(Z)\right].\end{aligned}$$
This beeing true for every function g, we can deduce (why?) that, Z-almost surely
$$\mathbb{E}[f(X) \mid Z] = \mathbb{E}[f(Y) \mid Z].$$

For the second part, note that $(X, Z) \sim (Y, Z)$ implies $X \sim Y \sim \mu$. Then we compute, for every measurable bounded function φ,
$$\begin{aligned} & \mathbb{E}[g(Z)\varphi(X)] = \mathbb{E}[g(Z)\varphi(Y)] \\ \Leftrightarrow \quad & \mathbb{E}[h_1(X)\varphi(X)] = \mathbb{E}[h_2(Y)\varphi(Y)] \\ \Leftrightarrow \quad & \mathbb{E}[h_1(X)\varphi(X)] = \mathbb{E}[h_2(X)\varphi(X)].\end{aligned}$$
Thus $\mathbb{E}[(h_1 - h_2)(X)\varphi(X)) = 0$, and this beeing true for every function φ we can deduce (why?) the desired result. ∎

Exercice 6 Let $T_1, \ldots, T_n$ be *i.i.d.* integrable random variables, et let $T := \sum_{i=1}^n T_i$. Show that

$$\mathbb{E}[T \mid T_1] = T_1 + (n-1)\,\mathbb{E}[T_1] \qquad \text{and} \qquad \mathbb{E}[T_1 \mid T] = \frac{T}{n}.$$

Proof. For the first part, we compute

$$\mathbb{E}[T \mid T_1] = \mathbb{E}[T_1 \mid T_1] + \sum_{i=2}^n \mathbb{E}[T_i \mid T_1] = T_1 + \sum_{i=2}^n \mathbb{E}[T_i] = T_1 + (n-1)\,\mathbb{E}[T_1],$$

by i) and iv) of Exercice 3, and Exercice 4.

For the second part we can notice that by the symmetry property proven in Ex5, for every $1 \leq i, j \leq n$ one has,

$$\mathbb{E}[T_i \mid T] = \mathbb{E}[T_j \mid T],$$

and that

$$\sum_{i=1}^n \mathbb{E}[T_i \mid T] = \mathbb{E}[\sum_{i=1}^n T_i \mid T] = \mathbb{E}[T \mid T] = T.$$

The result follows. ■